Section 8.2: Integration by Parts

When you finish your homework, you should be able to…

- π Use the integration by parts technique to find indefinite integral and evaluate definite integrals
- π Use the tabular method to organize an integral requiring integration by parts
- π Recognize trends and establish guidelines for integrals requiring integration by parts

Warm-up:

1. Differentiate with respect to the independent variable.

a. $f(x) = \arcsin 5x$

b. $y = \ln(5x+1)$

c. $r(\theta) = \tan\theta$

2. Find the indefinite integral.

a. $\int \frac{\arctan x}{x^2+1} dx$

b. $\int \frac{(\ln x)^3}{x} dx$

c. $\int x\sqrt{5-x}\,dx$

3. Evaluate the definite integral.

$$\int_0^{\pi/8} \tan^2 6\theta \sec^2 6\theta \, d\theta$$

_________________________ by _________________ is based on the formula for the _______________________ of a ______________________ and is useful for ___________________________ involving products of algebraic and ______________________________ functions.

Consider the following product of two functions of x that have continuous

________________________.

THEOREM: INTEGRATION BY PARTS

If u and v are functions of x and have ______________________ derivatives, then

This technique turns a super complicated integral into ______________ ones. The trick is to choose your function ____ so that __________ is __________________ than ____. Oh yeah…and **PRACTICE A _______________ OF PROBLEMS!!!**

Okay…let's look at the "_________" types of Integration by Parts (IBP) problems.

Which types of expressions do not have _________________ integration formulas?

1.

2.

EXAMPLE 1: Find the following indefinite integrals.

a. $\int 4x^2 \ln x dx$

b. $\int \arcsin x dx$

When you don't have a ___________________________ factor, you need to play around with the ______________________. Oftentimes it works out to let ______ be the factor whose ____________________ is a simpler __________________ than u. Then _______ would be the more _____________________ remaining factor. Use ______________________!!!

There is a lot of _______________ and _________________--especially at first☺

****Remember: _____ ALWAYS includes _______!**

EXAMPLE 2: Find the indefinite integral.

a. $\int \frac{6x}{e^{7x}} dx$

b. $\int x\sqrt{5-x}dx$

c. $\int x\sec^2 xdx$

Sometimes, you need to use IBP ____________________ times. You may even need to ____________________ like ____________________ (yes, you can do that)!

EXAMPLE 3: Find the indefinite integral.

a. $\int e^{-x} \cos 3x dx$

b. $\int x^2 e^{-x} dx$

THE TANZALIN (AKA TABULAR) METHOD is a way of organizing an integration by parts problem. Let's rework the last example using this method.

DERIVATIVES	**INTEGRALS**	**ALTERNATE SIGNS**	**SAME-COLOR PRODUCTS**
u			
	dv		

Let's try to bring this all together…

In general, use the following choice for u, in order.

1.

2.

3.

**When you have $\int e^{ax}\cos bx dx$ or $\int e^{ax}\sin bx dx$, let _____________ and let _______________ or let _______________.

**To evaluate a definite integral, first find the _______________ integral and then back substitute.

EXAMPLE 5: Find the indefinite integral or evaluate the definite integral.

a. $\int_0^1 x\arcsin x^2 dx$

b. $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx$

Section 8.3: Trigonometric Integrals

When you finish your homework, you should be able to…

- π Find indefinite integrals and evaluate definite integrals involving the sine and cosine functions which are raised to positive powers
- π Find indefinite integrals and evaluate definite integrals involving the secant and tangent functions which are raised to positive powers
- π Use trigonometric identities to find indefinite integral and evaluate definite integrals involving the sine and cosine functions

Warm-up 1: Simplify.

a. $1-\sin^2 x$

b. $1+\tan^2 x$

c. $\dfrac{1-\cos 2x}{2}$

d. $\dfrac{1+\cos 2x}{2}$

Warm-up 2: Complete the statement.

a. If $u=\sin 2x$, then $du=$______________.

b. If $u=\cos 4x$, then $du=$______________.

c. If $u=\tan x$, then $du=$______________.

d. If $u=\sec 6x$, then $du=$______________.

EXAMPLE 1: Find the indefinite integral.

a. $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}dx$

b. $\int \sin^3 x \cos^2 x dx$

So we discovered that if the sine portion of the integrand has an _______, positive integer as a power and the cosine portion has any other power, then we save ___________ sine factor, and _______________ the others to ______________ factors. Then ____________ and _______________.

c. $\int \sin^4 2x \cos^3 2x dx$

So we discovered that if the _______________ portion of the integrand has an odd, positive integer as a power and the sine portion has any other power, then we save _____________ cosine factor, and _________________ the others to ________________ factors. Then _______________ and ____________________.

d. $\int \sin^4 5x dx$

So we discovered that if only one sine or cosine factor is in the integrand and has an _______, positive integer as a power, you use the _________________ _________________ formula

______________________ or ______________________ until you can use basic integration formulas. What should we do if both the sine and cosine are raised to even, positive powers?

e. $\int \sec^4 x \tan^3 x dx$

So we discovered that if the ______________ portion of the integrand has an even, positive integer as a power and the tangent portion has any other exponent, then we save _______ ____________ factors, and convert the rest to _______________ factors. Then ______________ and __________________.

f. $\int \sec^3 5x \tan^3 5x dx$

So we discovered that if the ______________ portion of the integrand has an odd, positive integer as a power and the secant portion has any other exponent, then we save a ____________-____________ factor, and convert the rest to _______________ factors. Then ____________ and _______________.

g. $\int \tan^4 x dx$

So we discovered that if there is only a ___________ factor raised to a positive, ____________ power, rewrite as two ______________ factors, one of which is _______________, convert the other to ______________ ____________ minus _______, and then _______________ and ____________________.

h. $\int \sec^3 x dx$

So we discovered that if there is only a ____________ factor raised to a positive, _____________ power, we need to use _____________________ by __________.

If none of these techniques work, try converting all factors to __________ and _______________ factors. Then play around with identities.

EXAMPLE 2: Find the indefinite integral.

a. $\int \frac{\tan^2 x}{\sec^5 x} dx$

b. $\int \frac{\sin^2 x - \cos^2 x}{\cos x} dx$

PRODUCT TO SUM IDENTITIES

If ______ ______________ ___________ occur in the integral, use the following identities.

$$\sin mx \sin nx = \underline{\hspace{8cm}}$$

$$\sin mx \cos nx = \underline{\hspace{8cm}}$$

$$\cos mx \cos nx = \underline{\hspace{8cm}}$$

You do not need to memorize these identities.

EXAMPLE 3: Find the indefinite integral.

$$\int \sin 7x \cos 4x dx$$

EXAMPLE 4: Find the area of the region bounded by the graphs of $y = \cos^2 x$, $y = \sin x \cos x$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{4}$.

Section 8.4: Trigonometric Substitution

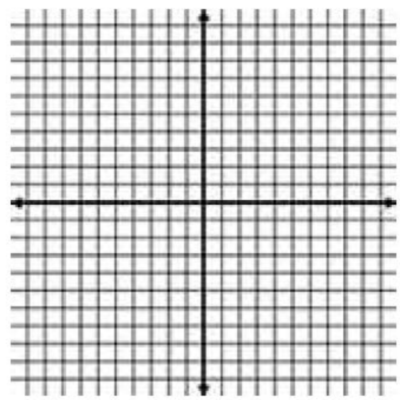
When you finish your homework, you should be able to…

- π Find indefinite integrals using trigonmometric substitution
- π Evaluate definite integrals using trigonmometric substitution

Warm-up 1: Consider the definite integral $\int_{-2}^{2} \sqrt{4-x^2}\,dx$. Do you have the

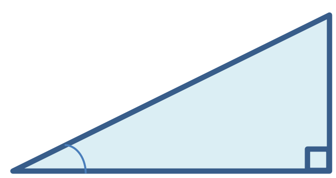
skills to evaluate this definite integral? ______________!

What tool did we use in Calculus I? ______________________!



Warm-up 2: Complete the figures.

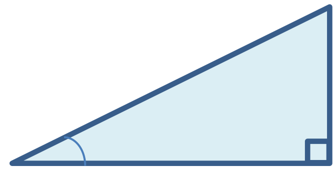
a. $u = a\sin\theta$



So, $\sqrt{a^2 - u^2} =$

for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

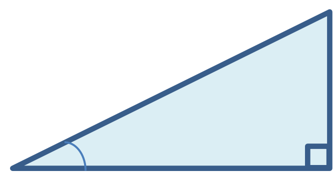
b. $u = a\tan\theta$



So, $\sqrt{a^2 + u^2} =$

for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

c. $u = a\sec\theta$



So, $\sqrt{u^2 - a^2} =$

$$\sqrt{u^2 - a^2} = \begin{cases} \rule{4cm}{0.4pt} & \text{for } u > a, \text{ where } 0 \le \theta < \dfrac{\pi}{2} \\ \rule{4cm}{0.4pt} & \text{for } u < -a, \text{ where } \dfrac{\pi}{2} < \theta \le \pi. \end{cases}$$

NOTE: These are the same intervals over which the ________________,

________________, and ________________________are defined. The

restrictions on ______ ensure that the function used for the substitution is

________-to-_______.

EXAMPLE 1: Evaluate the definite integral.

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

So we discovered that if the integrand has a ________________ and no basic integration rules, ___________, or regular __________________ integrals work, we use the substitution ____________________.

EXAMPLE 2: Find the indefinite integral.

a. $\int \frac{\sqrt{x^2 - 16}}{x} dx$

So we discovered that if the integrand has a _______________ and no basic integration rules, __________, or regular _________________ integrals work, we use the substitution ___________________.

b. $\int \frac{1}{x\sqrt{9x^2+1}}dx$

So we discovered that if the integrand has a ________________ and no basic integration rules, ___________, or regular __________________ integrals work, we use the substitution _____________________.

c. $\int \frac{x^2}{\sqrt{2x-x^2}}dx$

EXAMPLE 3: Evaluate the definite integral.

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-t^2\right)^{5/2}}\,dt$$

Section 8.5: Partial Fractions

When you finish your homework you should be able to…

- π Review how to decompose rational expressions into partial fractions
- π Utilize partial fractions to find indefinite integrals
- π Utilize partial fractions to evaluate definite integrals

Warm-up: Find the indefinite integral.

$$\int \frac{x^2 - x - 1}{x - 1} dx$$

(CASE 1) Q HAS ONLY NONREAPEATED LINEAR FACTORS

Under the assumption that Q has only __________________ linear factors, the polynomial Q has the form

where no two of the numbers ______________________ are equal. In this case, the partial fraction decomposition of _____________ is of the form

where the numbers _____________________ are to be determined.

Example 1: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{2}{9x^2-1}dx$$

(CASE 2) Q HAS REAPEATED LINEAR FACTORS

If the polynomial Q has a ____________________ linear factor, say

_____________, ______________, *n* is an ________________, then, in the

partial fraction decomposition of ____________, we allow for the terms

where the numbers _______________________ are to be determined.

Example 2: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{5x-2}{\left(x-2\right)^2}\,dx$$

(CASE 3) Q CONTAINS A NONREAPEATED IRREDUCIBLE QUADRATIC FACTOR

If Q contains a ____________________ irreducible quadratic factor of the form

_________________________, then, in the partial fraction decomposition of

_____________, allow for the term

where the numbers _______________________ are to be determined.

Example 3: Write the partial fraction decomposition of the rational expression in the integrand, and find the indefinite integral.

$$\int \frac{6x}{x^3-8}\,dx$$

(CASE 4) Q CONTAINS A REAPEATED IRREDUCIBLE QUADRATIC FACTOR

If the polynomial Q contains a ____________________ irreducible quadratic

factor of the form __________________________, _____________, *n* is an

_________________, then, in the partial fraction decomposition of __________,

allow for the terms

where the numbers _______________________________ are to be determined.

Example 4: Write the partial fraction decomposition of the rational expression in the integrand, and evaluate the definite integral.

$$\int_0^1 \frac{x^3}{\left(x^2+16\right)^2}\,dx$$

Example 5: Find the indefinite integral.

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\,dx$$

Section 8.7: Indeterminate Forms and L'Hôpital's Rule

When you finish your homework you should be able to…

- π Recognize all indeterminate forms
- π Apply L'Hôpital's Rule to evaluate limits
- π Manipulate expressions so that L'Hôpital's Rule may be applied to evaluate limits

WARM-UP: Find the limit. It is okay to write $\pm\infty$ as your answer.

1. $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}$

2. $\lim\limits_{x\to 1}\dfrac{x^3-x^2-x+1}{x^2-1}$

3. $\lim\limits_{\Delta x \to 0} \dfrac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x}$

4. $\lim\limits_{x \to \pi^{+}} \csc x$

5. $\lim_{x \to 0^+} \ln x$

6. $\lim_{x \to \infty} \arctan x$

7. $\lim_{x \to 0} \frac{\sin 4x}{x}$

8. $\lim_{x \to 0} \frac{1 - \cos^2 x}{x + x\cos x}$

What indeterminate form did you encounter in some of these problems?

What skills did you use to get these expressions into a determinate form?:

1.

2.

3.

L'Hôpital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possible at a).

Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

meaning that we have an _______________________ form of _____ or _____.

Then

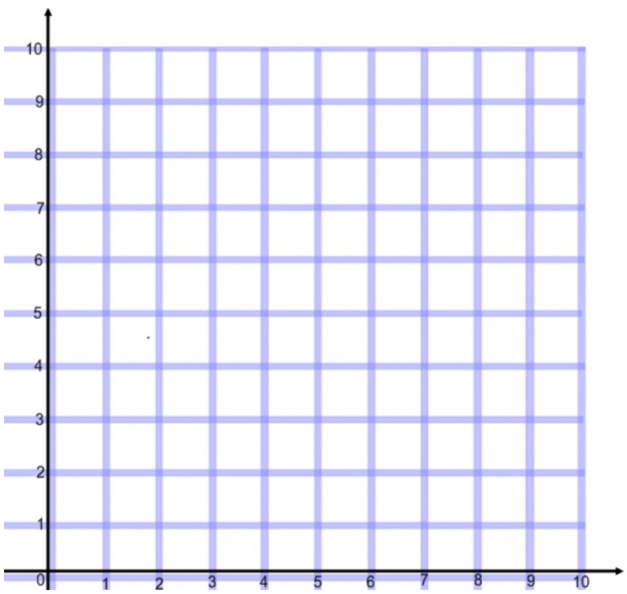
If the limit on the right side _____________ or is _________ or __________.

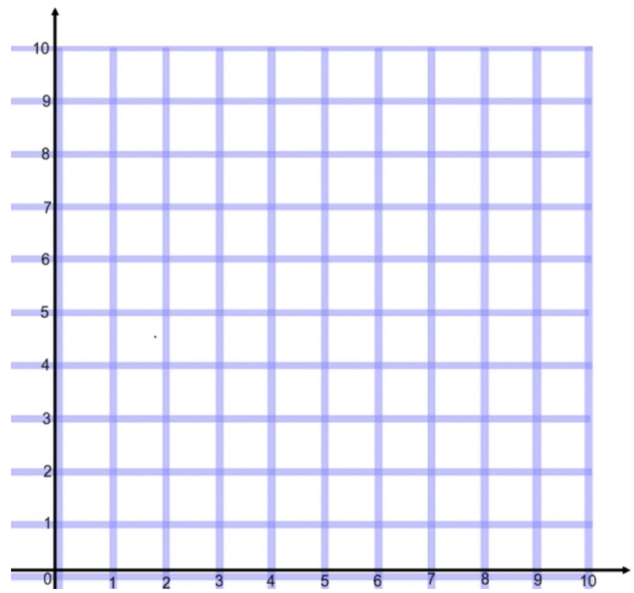
***What should we check before applying L'Hôpital's Rule?**

1. _______ and _______ are __________________ near _____ and
2. ___________ near ____.

****L'Hôpital's Rule is also valid for _____________________ limits and for limits at _______ or ________.**

*****Let's look at the special case when $f(a)=g(a)=0$, f' and g' are continuous, and $g'(x)\neq 0$.**





Example 1: Determine if L'Hôpital's Rule can be used to evaluate the limit. If so, apply L'Hôpital's Rule to evaluate the limit.

a. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

b. $\lim_{x\to 1}\frac{x^3-x^2-x+1}{x^2-1}$

c. $\lim_{x\to 0}\frac{\sin 4x}{x}$

d. $\lim\limits_{x \to 0} \dfrac{1-\cos^2 x}{x + x\cos x}$

e. $\lim\limits_{x \to \infty} \dfrac{\ln x}{\sqrt{x}}$

Indeterminate Forms

We already know that ________ and ________ represent 2 types of ________________ forms. There are also indeterminate _____________, __________________, and __________________.

Indeterminate Products occur when the limit of 1 ___________ approaches ______ and the other factor approaches ________ or ________. Suppose

___________________ and _________________. If ____ prevails, the result of the limit of the _______________ will be _____. If ______ is the victor, the ___________ of the product will be _____. If they decide to sign a ______________, the answer will be some ____________, _____________ number. To find out, see if you can ______________ the difference into a ______________.

Example 2: Evaluate the limit.

a. $\lim\limits_{x\to\infty} \sqrt{x}e^{-x/2}$

b. $\lim\limits_{x\to 0^+} \sin x \ln x$

Indeterminate Differences occur when both limits approach ______.

Suppose___________________ and __________________. If ____ prevails, the result of the limit of the _______________ will be _____. If ______ is the victor, the __________ of the product will be _____. If they decide to sign a ______________, the answer will be some ____________ number. To find out, see if you can ______________ the difference into a ______________ by using a ______________ ________________, __________________, or _____________ out a _____________ _______________.

Example 3: Evaluate the limit.

a. $\lim\limits_{x \to 0} (\csc x - \cot x)$

b. $\lim\limits_{x \to 1^+} \left[\ln\left(x^7 - 1\right) - \ln\left(x^5 - 1\right) \right]$

Indeterminate Powers occur when __________________. There are 3 indeterminate forms that arise from this type of limit.

1. ________________ and __________________. This yields the inderminate form ____________.
2. ________________ and __________________. This yields the inderminate form ____________.
3. ________________ and __________________. This yields the inderminate form ____________.

To find these types of limits, see if you can take the ______________ __________________:

or write the function as an ____________________:

Example 4: Evaluate the limit.

a. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

b. $\lim_{x\to 0^+} x^{\sqrt{x}}$

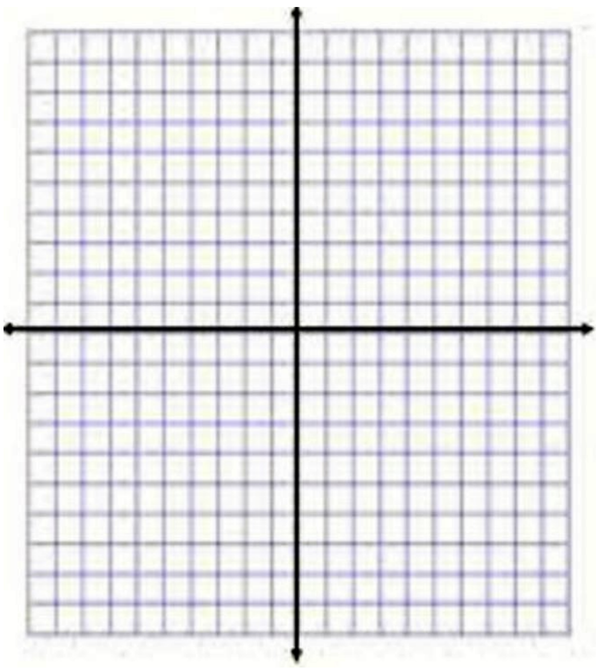
Section 8.8: Improper Integrals

When you finish your homework you should be able to…

- π Recognize when a definite integral is improper
- π Use your integration and limit techniques to evaluate improper integrals

WARM-UP: Consider the function $f(x)=\dfrac{2}{x^3}$.

1. Graph the function.



2. Find the limits. It is okay to write $\pm\infty$ as your answer.

   a. $\lim\limits_{x\to 0^+}\dfrac{2}{x^3}$

   b. $\lim\limits_{x\to\infty}\dfrac{2}{x^3}$

3. Evaluate the definite integral.

$$\int_1^b \frac{2}{x^3}\,dx$$

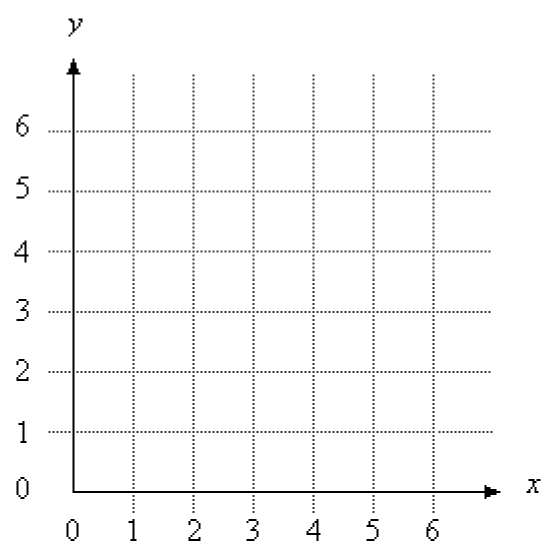
Let's put some stuff together☺

Recall the if a function is ______________________ on the interval ________, the ___________________ integral is equal to the ___________ under the _____________ and bounded by the ________________. Also remember that a function is said to have an infinite ____________________ at ____ when, from the __________ or the left,

TYPE 1: INFINITE INTERVALS

Now consider the following definite integral.

$$\int_1^\infty \frac{2}{x^3}\,dx$$



Definition: Improper Integrals With Infinite Integration Limits

1. Suppose f is continuous on the interval from __________, then

$$\int_a^{\infty} f(x)\,dx =$$

2. Suppose f is continuous on the interval from __________, then

$$\int_{-\infty}^{b} f(x)\,dx =$$

3. Suppose f is continuous on the interval from __________, then

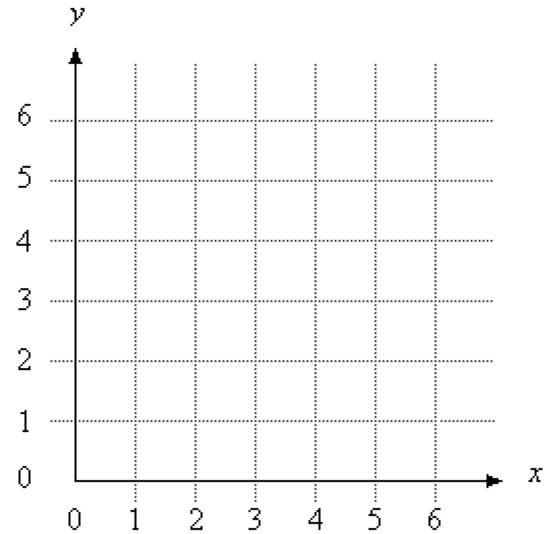
$$\int_{-\infty}^{\infty} f(x)\,dx =$$

where _____ is any real number.

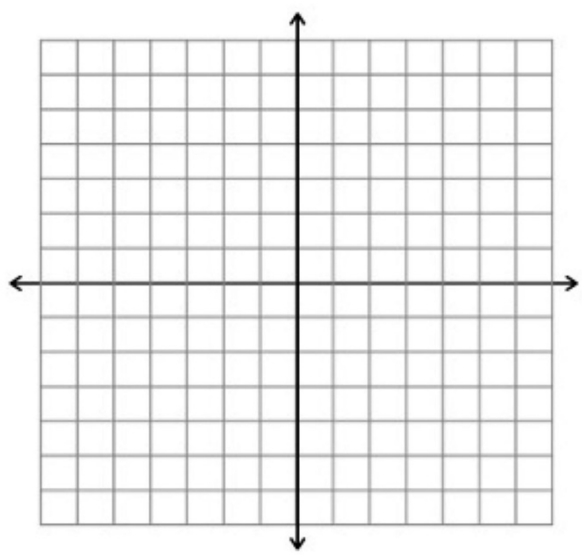
In the first two cases, the improper integral _________________ when the _____________ exists; otherwise the improper integral ________________. In the third case, the improper integral on the left __________________ when either of the improper integrals on the _______________ diverge.

Example 1: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_1^\infty \frac{dx}{x}$



b. $\int_{-\infty}^\infty \frac{dx}{4+x^2}$



TYPE 2: DISCONTINUOUS INTEGRANDS

Now consider the following definite integral.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx$$

Definition: Improper Integrals With Infinite Discontinuities

1. Suppose f is continuous on the interval from __________, and has an infinite _________________ at ____, then

   $$\int_a^b f(x)\,dx =$$

2. Suppose f is continuous on the interval from __________, and has an infinite _________________ at ____, then

   $$\int_a^b f(x)\,dx =$$

3. Suppose f is continuous on the interval from __________, except for some ______ in ____________ at which f has an infinite _________________ at ____, then

   $$\int_a^b f(x)\,dx =$$

In the first two cases, the improper integral _________________ when the _____________ exists; otherwise the improper integral ________________. In the third case, the improper integral on the left __________________ when either of the improper integrals on the _______________ diverge.

Example 2: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_{3}^{6} \frac{dx}{\sqrt{36-x^2}}$

b. $\int_{1}^{\infty} \frac{dx}{x \ln x}$

Consider $\int \frac{dx}{x^p}$, where *p* is a real number. Let's find the indefinite integral on

1. $p = 0$
2. $p \neq 1$
3. $p = 1$

Example 3: Determine all values of *p* for which the improper integral converges.

$$\int_1^{\infty} \frac{dx}{x^p}$$

THEOREM: A SPECIAL TYPE OF IMPROPER INTEGRAL

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \text{diverges}, & p < 1 \end{cases}$$

Example 4: If possible, evaluate the following definite integrals and ascertain if they are convergent or divergent.

a. $\int_1^\infty \frac{dx}{x^{1/2}}$

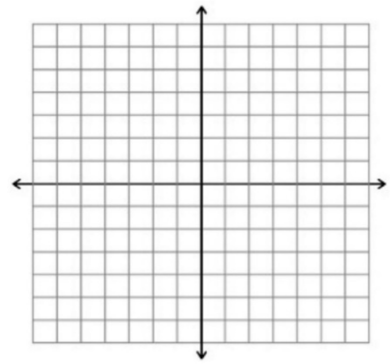
b. $\int_1^\infty x^{-3}dx$

9.1: Sequences

When you finish your homework you should be able to…

- Identify the terms of a sequence, write a formula for the *n*th term of a sequence, and ascertain whether a sequence converges or diverges.
- Use properties of monotonic sequences and bounded sequences.

WARM-UP: Consider the function $f(x) = \sqrt{x}$.



1. Sketch the graph of the function.

2. Find the following:

a. $f(0) =$ _____ c. $f(2) =$ _____ e. $f(4) =$ _____

b. $f(1) =$ _____ d. $f(3) =$ _____ f. $f(5) =$ _____

g. $\lim_{x \to 0} \sqrt{x} =$ _______ i. $\lim_{x \to 4} \sqrt{x} =$ _______

h. $\lim_{x \to 1} \sqrt{x} =$ _______ j. $\lim_{x \to \infty} \sqrt{x} =$ _______

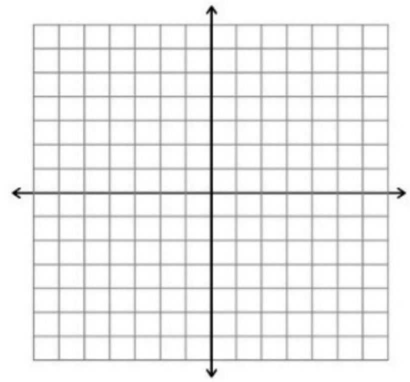
3. Now consider $a_n = \sqrt{n}$.
4. Find the following:

a. $a_1 =$ _____ c. $a_3 =$ _____ e. $a_5 =$ _____

b. $a_2 =$ _____ d. $a_4 =$ _____

Hmmm…so it looks like _______ equals _______ at all of the _____________ _____________.

5. Sketch the graph of the sequence.



Definition of the Limit of a Sequence

Let L be a real number. The ________________ of a sequence _______ is _____, written as

$$\lim_{n\to\infty} a_n = L$$

if for $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L exists, then the sequence ____________________. If the limit does not exist, then the sequence ____________________.

Looking at the two graphs we sketched, as ____________ it looks like ____________. So, we would say the $\lim\limits_{n\to\infty} a_n$ ________. and ________ ____________.

EXAMPLE 1: Write the first five terms of the sequence.

a. $a_n = \dfrac{3n}{n+4}$

b. $a_1 = 6,\ a_{k+1} = \dfrac{1}{3}a_k{}^2$

FACTORIALS are factors which decrease by one. So 5!, read as "five factorial" is

5! = ____________________ = _______. We will be working with unknown factorials.

In general, *n*! = ____________________________________, and 0! = _____.

EXAMPLE 2: Simplify the ratio of factorials.

a. $\dfrac{n!}{(n+2)!}$

b. $\dfrac{(2n+2)!}{(2n)!}$

EXAMPLE 3: Find the *n*th term of the sequence.

a. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

b. $\frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \frac{1}{5\cdot 6}, \ldots$

Theorem: Limit of a Sequence

Let *L* be a real number. Let f be a function of a real variable such that

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer *n*, then

EXAMPLE 4: Find the limit of the sequence, if it exists.

a. $a_n = 6 + \frac{2}{n^2}$

b. $a_n = \cos\frac{2}{n}$

Theorem: Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty}(a_n \pm b_n) =$ ______________
2. $\lim_{n\to\infty} ca_n =$ _____, c is any _________ number.
3. $\lim_{n\to\infty}(a_n b_n) =$ ______________
4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$ _____, ____________ and _________.

EXAMPLE 5: Determine the convergence or divergence of the sequence with the given *n*th term. If the sequence converges, find its limit.

a. $a_n = \frac{1+(-1)^n}{n^2}$

b. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

c. $a_n = \dfrac{(n-2)!}{n!}$

Absolute Value Theorem

For the sequence $\{a_n\}$, if

$\lim_{n\to\infty} |a_n| = 0$ then ____________________.

Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that________________,

then _____________________.

EXAMPLE 6: Show that the sequence converges and find its limit.

$$c_n = (-1)^n \frac{1}{n!}$$

Definition: Monotonic Sequence

A sequence $\{a_n\}$ is _________________ when its terms are _________________, $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$ or when its terms are nonincreasing _______________________________.

Definition: Bounded Sequence

1. A sequence $\{a_n\}$ is _________________ above when there is a real number _____ such that $a_n \le M$ for all _____. The number _____ is called an ___________ _____________ of the sequence.
2. A sequence $\{a_n\}$ is bounded _____________ when there is a real number _____ such that $N \le a_n$ for all _____. The number _____ is called a ___________ bound of the sequence.
3. A sequence ______ is ________________ when it is bounded _____________ and _____________ below.

Theorem: Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is ___________________ and ________________________, it _____________________.

EXAMPLE 7: Determine whether the sequence with the given *n*th term is monotonic and whether it is bounded.

$$a_n = \frac{\cos n}{n}$$

EXAMPLE 8: Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month?

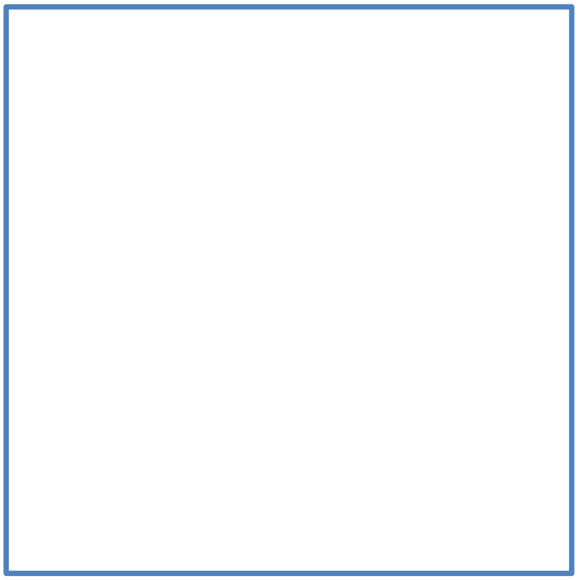
9.2: Series and Convergence

When you finish your homework you should be able to…

- π Understand and represent a convergent infinite series.
- π Use properties of infinite geometric series.
- π Use the *n*th term test for **divergence**.

We spent the last section checking out ____________________, and ascertaining whether a given sequence , a_n , ___________________ or ____________________ as ________________. Remember, the _____________ of a sequence are represented as a ______ or _________, which need not be ordered. There are finite and ________________ sequences. What if we were interested in ________________ a sequence? If we are interested in summing a finite number, say *n*, of the ______________ of a sequence, we would be finding the _____ _________________ _______. If we are interesting in finding the sum of an infinite sequence, if it exists, we would be finding an ______________ sum, called an infinite ____________, or just a ____________.

Our main interest will be to ascertain whether a series ________________ or ________________.



EXAMPLE 1: Consider the sequence we found above.

a. Write the first five terms, and the *n*th term of the sequence.

b. Sum the first five terms.

c. Represent this 5^{th} partial sum as a summation.

d. Find the limit of the sequence.

e. Find an expression for the *n*th partial sum.

f. What must the limit of this expression equal?

Definition: Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, The ____ ___________ sum is

If the sequence of partial sums $\{S_n\}$ ______________ to S , then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the _________ of the ____________.

If $\{S_n\}$ diverges, then the series __________________.

So from our first example, $S =$ ______________ , and this series

_________________ since the ____ ______________ sum ____________.

GEOMETRIC SERIES:

Theorem: Convergence of a Geometric Series

A geometric series with ________ ___ converges to the sum

when ___________________. Otherwise, for ____________, the series diverges.

EXAMPLE 2: Express the number as a ratio of integers.

$0.\overline{46}$

EXAMPLE 3: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and find its sum.

NOTE: The series in example 3 is called a ________________________ series.

EXAMPLE 4: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

NOTE: The series in example 4 is called a ________________________ series.

Theorem: Properties of Infinite Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then the following series converge to the indicate sums.

1. $\sum_{n=1}^{\infty} ca_n =$ ___
2. $\sum_{n=1}^{\infty} (a_n + b_n) =$ ____
3. $\sum_{n=1}^{\infty} (a_n - b_n) =$ ____

EXAMPLE 5: Determine the convergence or divergence of the series. If the series converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$$

Theorem: Limit of the *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then ____________________.

Theorem: *n*th Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ ________________________.

EXAMPLE 6: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \arctan n$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{4^n}$

c. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

EXAMPLE 7: Find all values of x for which the series converges. For these values of x, write the sum as a function of x.

$$\sum_{n=0}^{\infty} 5\left(\frac{x-2}{3}\right)^n$$

EXAMPLE 8: A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

9.3: The Integral Test, P-Series, and Harmonic Series

When you finish your homework you should be able to…

- π Use the Integral Test to ascertain whether an infinite series converges or diverges.
- π Determine whether a *p*-series converges or diverges.
- π Use properties of harmonic series.

WARM-UP: Determine whether the improper integral converges or diverges.

1. $\int_1^\infty \frac{\ln x}{x^3}\,dx$

2. $\int_1^{\infty} \frac{1}{3^x}\,dx$

3. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Theorem: The Integral Test

If f is _______________, ________________, and __________________ for $x \geq 1$ and $a_n = f(n)$, then

Either both _____________________ or both _____________________.

*****NOTE:** Our interest is whether the series converges or diverges as _______, so the index of the summation can start at some integer ____ as opposed to a___ when we apply the integral test.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

b. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1}$

P-Series and Harmonic Series

A harmonic series is the ____________ of sounds represented by _____________ waves in which the _______________ of each sound is an ______________ multiple of the _______________ frequency. Pythagoras and his students discovered this relationship between the _______ and the _________ of the vibrating string. The most beautiful harmonies seemed to correspond with the simplest ___________ of _________ numbers. Later mathematicians developed this idea into the ______________ series, where the ___________ in the harmonic series correspond to the node on a _________________ string that produce _________________ of the fundamental frequency. So, ___________ is ______________ the fundamental frequency, ___________ is ______________ times the fundamental frequency, and so on. In music, strings of the same ________________, ___________________, and ________________, and whose _________________ form a harmonic series, produce _______________ tones. A general harmonic series is of the form ___________________________________. The harmonic series is a special case of the ______________, where _________.

Theorem: Convergence of *p*-Series

The *p*-series

___________________ for __________, and ___________________ for ________________.

EXAMPLE 2: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

b. $1 + \frac{1}{\sqrt[5]{4}} + \frac{1}{\sqrt[5]{9}} + \frac{1}{\sqrt[5]{16}} + \frac{1}{\sqrt[5]{25}} + \cdots$

9.4: Series Comparison Tests

When you finish your homework you should be able to…

- π Use the Direct Comparison Test to ascertain whether an infinite series converges or diverges.
- π Use the Limit Comparison Test to ascertain whether an infinite series converges or diverges.

WARM-UP: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

Our Tests So Far…

***n*th Term Test for** ________________. If ____________, the series ________________. If ______________, we need to ______ further __________!!!

Geometric Series is of the form ____________. If ________________, the series _______________ and its _______ is ____________________. Otherwise, the series diverges.

Telescoping Series. Requires _______________ ______________ decomposition. The __________ is the sum of the terms which do not ____________ out plus _________________.

***p*-Series** is of the form _______________. If ___________, the series ___________________. If __________________, the series __________________.

The Integral Test requires that ____ is _______________, continuous, and _______________ for $x \geq 1$, and $f(n) = a_n$ for all n. If _____________ converges, ______________ converges. Otherwise, ___________ diverges.

Theorem: The Direct Comparison Test

Let ____________________ for all n.

1. If $\sum_{i=1}^{\infty} b_n$ ____________________, ______________ converges.
2. If $\sum_{i=1}^{\infty} a_n$ ____________________, $\sum_{i=1}^{\infty} b_n$ ___________________.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

b. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Theorem: The Limit Comparison Test

If ________, ________, and

where L is ____________ and ______________, then

$\sum_{i=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ either both ___________________ or both _________________.

NOTE: When choosing your comparison, you can disregard all but the

_________________ powers of _____. So, if we are testing $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{5n^2+2}$, our

comparison series would be _____________ = ___________.

Proof:

EXAMPLE 2: Determine the convergence or divergence of the series. Explain.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1}$

b. $\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

9.5: Alternating Series

When you finish your homework you should be able to…

- π Use the Alternating Series Test to ascertain whether an infinite series converges or diverges.
- π Use the Alternating Series Remainder to approximate the sum of an alternating series.
- π Classify a convergent series as conditionally convergent or absolutely convergent.

WARM-UP: Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2)^{n+1}}$$

Theorem: Alternating Series Test

Let __________________. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge when both conditions below are met.

1. __________________ 2. __________________, for all n.

NOTE: The second condition can be modified to require that ______________ for all ___ greater than some integer ____.

EXAMPLE 1: Determine the convergence or divergence of the series. Explain.

a. $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2)^{n+1}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

d. $\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi$

e. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{1\cdot 4\cdot 7\cdot 10\cdots(3n-2)}$

Theorem: Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the

________________ value of the _______________ ______ involved in

approximating the sum ____ by ____ is less than or equal to the first

________________ term.

EXAMPLE 2: Approximate the sum of the series by using the first six terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$$

EXAMPLE 3: Determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}$$

Theorem: Absolute Convergence

If the series __________ converges, then the series _________ also converges.

Which of our examples would be an example of this theorem?

Definition of Absolute and Conditional Convergence

1. The series $\sum a_n$ is ____________________ convergent when _________ converges.

2. The series $\sum a_n$ is ____________________ convergent when _______ converges but _________ diverges.

EXAMPLE 4: Determine whether the series converges absolutely or conditionally, or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\arctan n$

c. $\sum_{n=1}^{\infty} \frac{\sin\left[\left(2n+1\right)\frac{\pi}{2}\right]}{n}$

9.6: The Ratio and Root Tests

When you finish your homework you should be able to…

- π Use the Ratio Test to ascertain whether an infinite series converges or diverges.
- π Use the Root Test to ascertain whether an infinite series converges or diverges.
- π Review Tests for convergence and divergence of an infinite series.

Theorem: The Ratio Test

Let $\sum a_n$ be a series with ____________________ terms.

1. The series $\sum a_n$ converges ________________ when ____________________.
2. The series $\sum a_n$ diverges when _________________ or _________________.
3. The Ratio Test is _________________ when ___________________.

EXAMPLE 1: Determine the convergence or divergence of the series using the Ratio Test.

a. $\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^{n}$

b. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

d. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

Theorem: The Root Test

1. The series $\sum a_n$ converges _______________ when _________________.

2. The series $\sum a_n$ diverges when ________________ or ________________.

3. The Root Test is ________________ when _________________.

EXAMPLE 2: Determine the convergence or divergence of the series using the Root Test.

a. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

b. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{5n+1}\right)^{n}$

c. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^{n}$

d. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

NOW IT'S UP TO YOU!!! DETERMINE WHETHER THE FOLLOWING INFINITE SERIES CONVERGE OR DIVERGE

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{2\cdot 5\cdot 8\cdot 11\cdots(3n-1)}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

3. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+3n}}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n n!(2n-1)}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

5. $\sum_{n=1}^{\infty}\left(\frac{n}{500}\right)^{n}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

6. $\sum_{n=1}^{\infty} e^{-4n}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

7. $\sum_{n=1}^{\infty} \frac{5^n - 1}{6^n - 1}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

8. $\sum_{n=1}^{\infty} \arctan n$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

9. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

10. $$\sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

Step 1: Identify the test(s) and conditions (if applicable).

Step 2: Run the test.

Step 3: Conclusion.

9.7: Taylor Polynomials

When you finish your homework you should be able to…

- π Find Taylor and Maclaurin polynomial approximations of elementary functions.
- π Use the remainder of a Taylor polynomial.

Some uses of the Taylor series for analytic functions include:

- The _____________ _______ of the series can be used as ______________________ of the entire function. Keep in mind that you need a sufficient amount of ______________.
- ______________________ and ______________________ of power series is _____________________ since it can be done ____________ by term.
- ___________________ operations can be done on the _______________ series ______________________. For example, ________________ formula follows from Taylor series __________________ for _________________ and ____________________ functions. This result is important in the field of _________________ analysis.
- _______________________ using the first few terms of a Taylor series can make otherwise _______________ problems possible for a restricted

domain. This is often used in _______________.

To find a ___________________ function ___ that ___________________ another function ___, we choose a number ___ in the ______________ of ___ at which _______________. This approximating ________________ is said to be _________________ about ___ or ________________ at ___. The evil plan is to find a polynomial whose ___________ looks like the graph of ____ ___________ this point. If we require that the ___________ of the polynomial function is the __________ as the slope of the __________ at ____, then we also have _________________. Using these two requirements we can get a ______________ approximation of ____.

EXAMPLE 1: Consider $f(x)=\dfrac{x}{x+1}$.

a. Find a first-degree polynomial function $P_1(x)=a_0+a_1x$ whose value and slope agree with the value and slope of f at $x=0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\dfrac{x}{x+1}$							
$P_1(x)$							

b. Now find a second-degree polynomial function $P_2(x) = a_0 + a_1 x + a_2 x^2$ whose value and slope agree with the value and slope of f at $x = 0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\frac{x}{x+1}$	-4	-0.25	-0.1111	0	0.0909	0.16667	0.5
$P_2(x)$							

c. Let's go for a third-degree polynomial function $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ whose value and slope agree with the value and slope of f at $x = 0$.

x	-0.8	-0.2	-0.1	0	0.1	0.2	1.0
$\frac{x}{x+1}$	-4	-0.25	-0.1111	0	0.0909	0.16667	0.5
$P_3(x)$							

Definition of *n*th Taylor and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

is called the ____ ________________ polynomial for ___ at ___.

If ________, then

is also called the ____ __________________ polynomial for ____.

Remainder of a Taylor Polynomial

To __________________ the ___________________ of approximating a function

value _____ by the Taylor polynomial ______, we use the concept of a

____________________ _______.

EXAMPLE 2: Consider the function $f(x)=x^2\cos x$.

a. Find the second Taylor polynomial for the function $f(x)=x^2\cos x$ centered at π.

b. Approximate the function at $x=\frac{7\pi}{8}$ using the polynomial found in part a.

Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

where

A ___________________ of this theorem is that

where _________________ is the __________________ value of ____________

between ____ and ___.

For _________ we have

Does this look familiar?

EXAMPLE 3: Use Taylor's Theorem to obtain an upper bound for error of the approximation. Then calculate the exact value of the error.

$$e \approx 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\frac{1^4}{4!}+\frac{1^5}{5!}$$

EXAMPLE 4: Determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of *x* to be less than 0.001.

$\cos(0.1)$

9.8: Power Series

When you finish your homework you should be able to…

- π Find the radius and interval of convergence of a power series.
- π Determine the endpoint convergence of a power series.
- π Differentiate and integrate a power series.

WARM-UP: Find the sixth-degree Maclaurin polynomial for $f(x)=e^x$.

This enables us to be able to ________________________ the function

_________ near ________. We found out that the higher the __________ of the

approximating ____________________, the better the approximation becomes.

In this section, you'll see that several important ___________________ can be

represented ____________________ by ________________ series.

Definition of Power Series

If x is a variable, then an infinite series of the form

is called a _____________ series _______________ at _____, where ___ is a constant.

If a power series is ______________ at ____, the power series will be of the form

EXAMPLE 1: Find the power series for $f(x) = e^x$, centered at $x = 0$.

Radius and Interval of Convergence

A power series in ____ can be thought of as a ______________ of ____.

The _______________ of ___ is the _______ of all ___ for which the power series __________________. Every power series converges at its __________.

Therefore, _____ is always in the _______________ of ____. The domain of a power series can take on any one of the following forms:

a ______________ ____________

an ________________

the _______ of ________ numbers

Theorem: Convergence of a Power Series

For a power series centered at c, precisely one of the following is true:

1. The series converges only at ______.
2. There exists a ________ number _______ such that the series converges
   _____________________ for ____________ and diverges for
   ______________.
3. The series converges absolutely for ______ ______.

Endpoint Convergence

Each __________________ must be _____________ for ______________ or _________________. This results in _____ possible forms an _____________ of _________________ can take on.

_________ 0  _________ _____ 

_________ ____

Example 2: Find the radius and interval of convergence (including a check for convergence at the endpoints) of the following power series.

a. $\sum_{n=0}^{\infty}(2x)^n$

b. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

c. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

Theorem: Properties of Functions Defined by Power Series

If the function ____

has a radius of convergence of _______, then, on the interval ______________

f is __________________ and thus _______________. The derivative and antiderivative are given below:

1.

2.

The radius of convergence of the series obtained by __________________ or ______________________ a power series is the ___________ as that of the ___________________ power series. What may change is the _______________ of convergence.

Example 3: Let $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$.

a. Find the interval of convergence of f.

b. Find the interval of convergence of g.

c. Show that $f'(x) = g(x)$.

d. Show that $g'(x) = -f(x)$.

e. Identify the function f.

f. Identify the function g.

Example 4: Write an equivalent series with the index of summation beginning at $n = 1$.

a. $\sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

9.9: Representing Functions as Power Series

When you finish your homework you should be able to…

- π Manipulate a geometric series to represent a function as a power series
- π Differentiate or integrate a geometric series to represent a function as a power series.

WARM-UP: Find the infinite sum of the convergent series $\sum_{n=0}^{\infty} 5\left(-\frac{3}{4}\right)^n$.

Now consider the function $f(x)=\frac{1}{1-x}$.

This __________________represents $f(x)=\dfrac{1}{1-x}$ only on the interval from

_______________. What is the domain of f ? _____________________.

How would we represent f on another interval? We must develop a

________________ _____________ which is ______________ at a different

value.

Example 1: Find the power series for $f(x)=\dfrac{1}{1-x}$ centered at $c=-2$.

Example 2: Find a geometric power series for the function $f(x) = \dfrac{2}{5-x}$ centered at 0, (a) by manipulating the function into the format of a geometric power series and (b) by using long division.

Example 3: Find a power series for the function, centered at c, and determine the interval of convergence.

a. $f\left(x\right)=\dfrac{3}{2x-1},\ c=2$

b. $f\left(x\right)=\dfrac{4}{3x-2},\ c=3$

Operations with Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ be power series centered at 0.

1. $f(kx)=\sum_{n=0}^{\infty} a_n k^n x^n$, where _____ is a _______________.
2. $f\left(x^N\right)=\sum_{n=0}^{\infty} a_n x^{nN}$, where _____ is a _______________.
3. $f(x)\pm g(x)=\sum_{n=0}^{\infty}\left(a_n \pm b_n\right)$

Note: These operations can change the ______________ of _____________ for the resulting series.

Example 4: Find a power series for the function, centered at *c*, and determine the interval of convergence.

a. $f(x)=\dfrac{5}{5+x^2}$, $c=0$

b. $f\left(x\right)=\dfrac{3x-8}{3x^2+5x-2},\ c=0$

Example 5: Consider the functions $f\left(x\right)=\dfrac{1}{1+x}$ and $g\left(x\right)=\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^{n}$.

a. Find a power series for f, centered at 0.

b. Use your result from part a to determine a power series, centered at 0, for the function $h\left(x\right)=\dfrac{x}{x^{2}-1}=\dfrac{1}{2\left(1+x\right)}-\dfrac{1}{2\left(1-x\right)}$. Identify the interval of convergence.

c. Use your result from part a to determine a power series, centered at 0, for the function $r(x)=\frac{2}{(x+1)^3}$. Identify the interval of convergence.

d. Use your result from part a to determine a power series, centered at 0, for the function $s(x)=\ln(1-x^2)$. Identify the interval of convergence.

9.10: Taylor and Maclaurin Series

When you finish your homework you should be able to…

- π Find a Taylor series or a Maclaurin series for a function.
- π Find a binomial series.
- π Use a basic list of Taylor series to derive other power series.

WARM-UP: Find the 8th degree Maclaurin polynomial for the function $f(x) = \cos x$.

Now let's see if we can form a power series!

What about that interval of convergence?

Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x)=\sum a_n(x-c)^n$ for all x in an open interval I containing c, then

and

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

is called the ________________ series for ______ at ____. If _________,

then the series is the ___________________ series for _____.

Example 1: Find the Taylor series, centered at c, for the function.

a. $f(x) = e^{-4x},\ c = 0$

b. $f(x)=\dfrac{1}{1-x},\ c=2$

Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$.

Example 2: Prove that the Maclaurin series for $f(x) = \cos x$ converges to $f(x)$ for all x.

Binomial Series

Let's check out the function $f(x)=(1+x)^k$, where k is a rational number. What do you think the Maclaurin series is for this function? Guess what…YOU KNOW HOW TO FIND IT!!! So, on your mark, get set, GO!

1. ________________ $f(x)$ a bunch of times and evaluate each

   ________________ at ______. Evil plan: __________________ a

   ___________________.

2. Determine the _______________ of _________________...Don't forget

to test the _________________!

Guidelines for Finding a Power Series

1. ____________________ $f(x)$ and __________________ each

   ____________________ at ______ until you find a ______________________.

2. Form the _______________ coefficient ____________________, and

   determine the ________________ of convergence for the ______________

   series.

3. Determine whether the series ___________________ to _________ within

   the interval of convergence.

Example 3: Find the Maclaurin series for the function using the binomial series.

a. $f(x) = \dfrac{1}{(1+x)^4}$

b. $f\left(x\right)=\sqrt{1+x^3}$

A Basic List of Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x} =$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x =$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x =$	$-1 \le x \le 1$
$\arcsin x =$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$	$-1 < x < 1$*

*convergence at endpoints depends on k

Example 4: Find the Maclaurin series for the function using the basic list of power series for elementary functions.

a. $f(x) = \ln(1 + x^2)$

b. $f(x) = e^x + e^{-x}$

c. $f\left(x\right)=\cos^2 x$

d. $f(x) = x\cos x$

Example 5: Find the first four nonzero terms of the Maclaurin series for the function $f(x) = e^x \ln(1+x)$.

Example 6: Use a power series to approximate the value of the integral with an error less than 0.0001.

$$\int_0^{1/2} \arctan x^2 dx$$

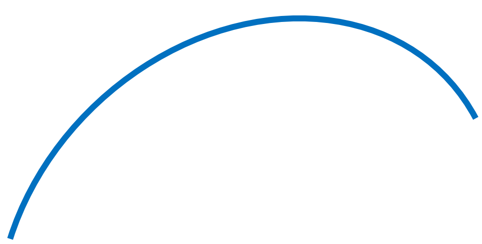
7.4: Arc Length and Surfaces of Revolution

When you finish your homework you should be able to…

- π Find the arc length of a smooth curve.
- π Find the area of a surface of revolution

Arc length is approximated by _____________ infinitely many _______________.

A ________________ curve is one which has a ___________ arc length. A sufficient condition for the graph of a function ____ to be rectifiable between ____________ and ______________ is that _____ be continuous on _____. A function of this type is considered to be ____________________ differentiable on ________ and its graph on the interval _______ is a _________ ________.



Definition of Arc Length

Let the function ____________ represent a smooth curve on the interval ______.

The arc length of ____ between ___ and ___ is

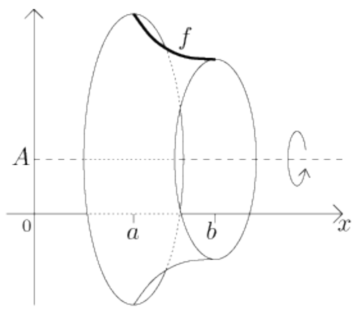
For a smooth curve __________ on the interval ______ the arc length of ____

between ___ and ___ is

EXAMPLE 1: Find the arc length from $(-3,4)$ clockwise to $(4,3)$ along the circle $x^2+y^2=25$. Show that the result is one-fourth the circumference of a circle.

Definition of Surface of Revolution

When the graph of a continuous function is _____________ about a ________,the resulting surface is a ___________ of ___________________.



Definition of the Area of a Surface of Revolution

Let the function ____________ have a continuous derivative on the interval _______. The area _____ of the surface of revolution formed by revolving the graph of ___ about a horizontal or vertical axis is

where ______ is the distance between the graph of ___ and the axis of revolution.

If ___________ on the interval ________ then the surface area is

where ______ is the distance between the graph of ___ and the axis of revolution.

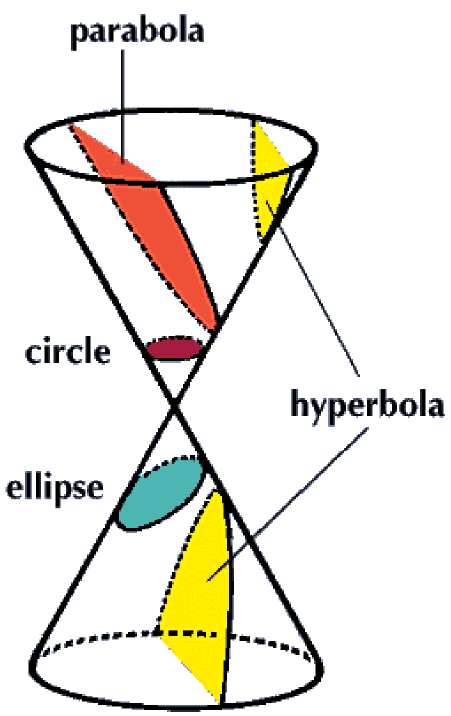
EXAMPLE 2: Find the area of the surface generated by revolving the curve $y = 9 - x^2$ about the *y*-axis.

10.1: Conics and Calculus

When you finish your homework you should be able to…

- π Use properties of conic sections to analyze and write equations of parabolas, ellipses, and hyperbolas.
- π Classify the graph of an equation of a conic section as a circle, parabola, ellipse, or hyperbola.
- π Find the equations of lines tangent and normal to conic sections

The graph of each type of ______________ section can be described as the intersection of a plane and two identical ___________ which are connected at their vertices.



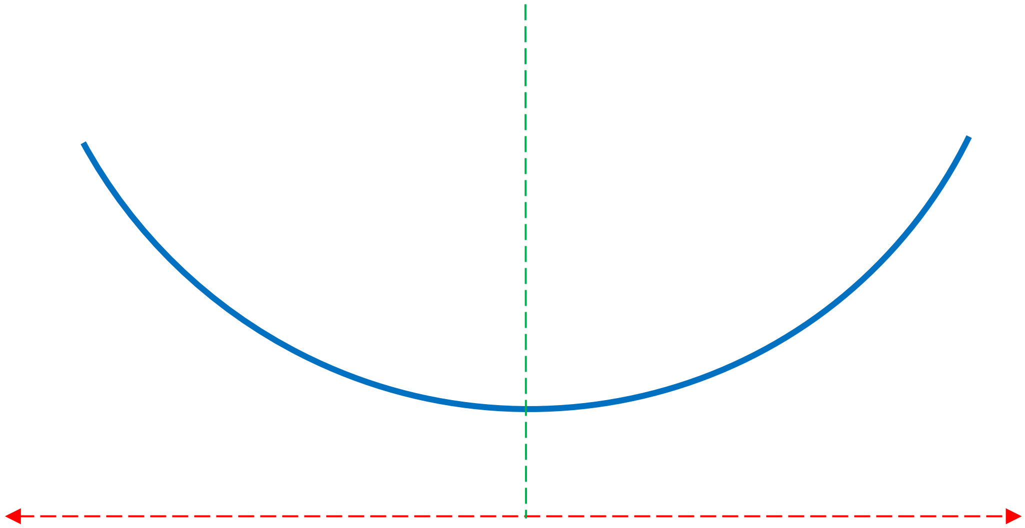
A **parabola** is the set of all

____________________ ___________ that are

_________________________ from a fixed line

called the _____________________ and a fixed

point called the _____________________.



Theorem: Standard Equation of a Parabola

The standard form of a parabola with vertex __________ and directrix ________ is

Vertical axis

The standard form of a parabola with vertex __________ and directrix ________ is

Horizontal Axis

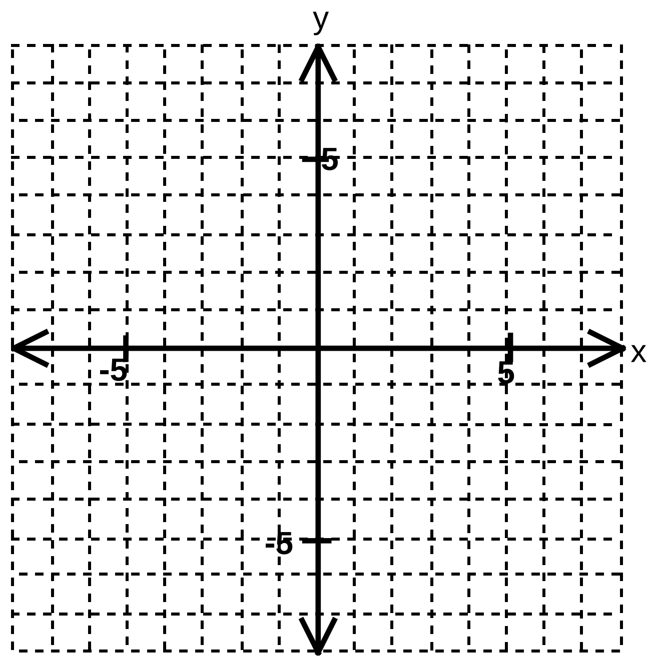
The focus lies on the axis ___ units from the vertex. The coordinates of the focus are

Vertical axis

Horizontal Axis

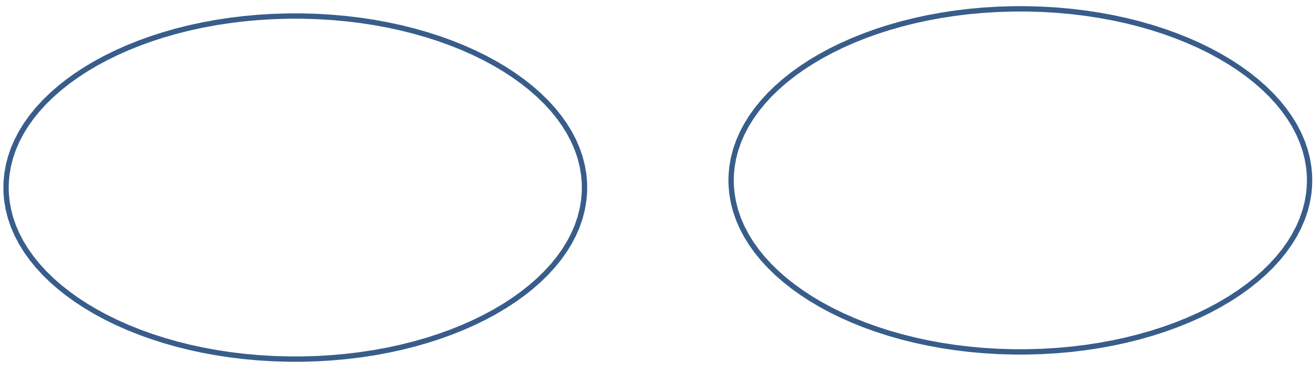
EXAMPLE 1: Consider $y^2+6y+8x+25=0$.

a. Find the vertex, focus, and the directrix of the parabola and sketch its graph.



b. Find the equation of the line tangent to the graph at $x=-4$.

An **ellipse** is the set of all __________________ _________ the sum of whose distances from two distinct fixed points called _______________ is constant.



Theorem: Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center _________ and major and minor axes of lengths ______ and ______, where _________, is

Major Axis is Horizontal

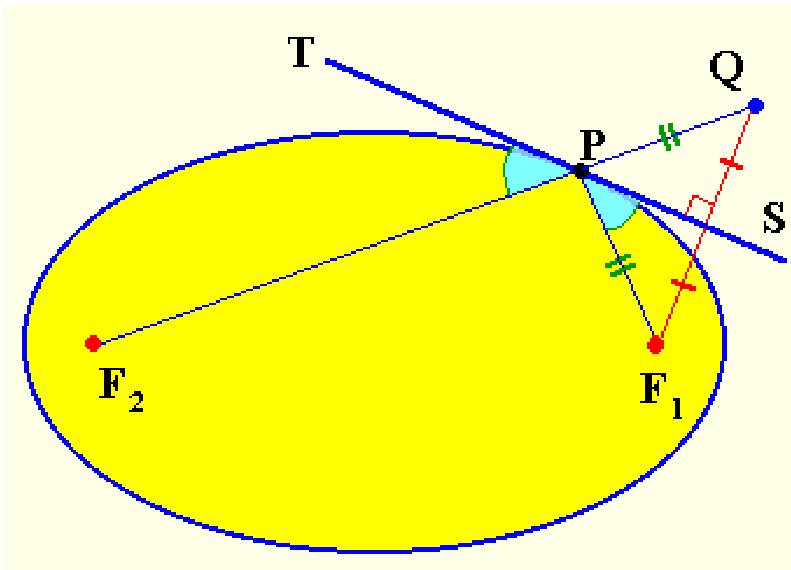
or

Major Axis is Vertical

The foci lie on the major axis, ___ units from the center, with

Theorem: Reflective Property of an Ellipse

Let ____ be a point on an ellipse. The tangent line to the ellipse at point ___ makes __________ angles with the lines through ____ and the _________.



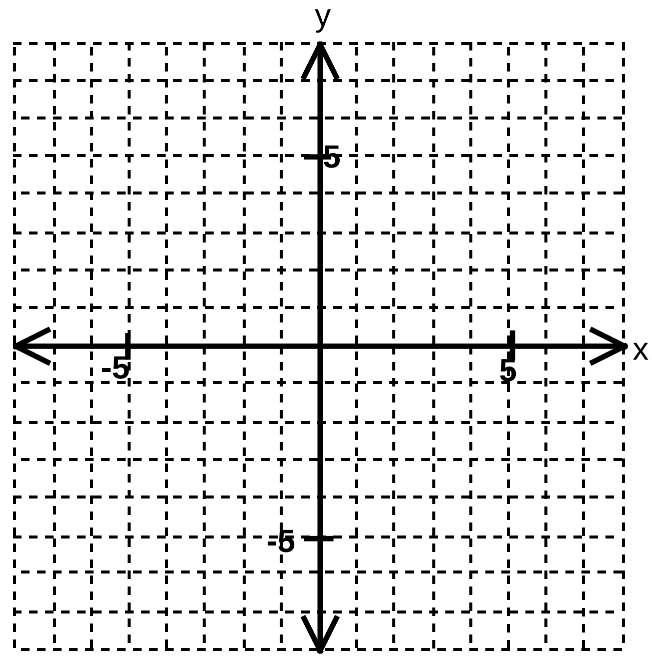
Definition of Eccentricity of an Ellipse

The ____________________ ____ of an ellipse is given by the ratio

For an ellipse that is close to being a _________________, the foci are close to the __________________ and the _______________ ___________ is close to _____________.An ___________________ ellipse has foci which are close to the _________________ and the ____________ ________ is close to ___.

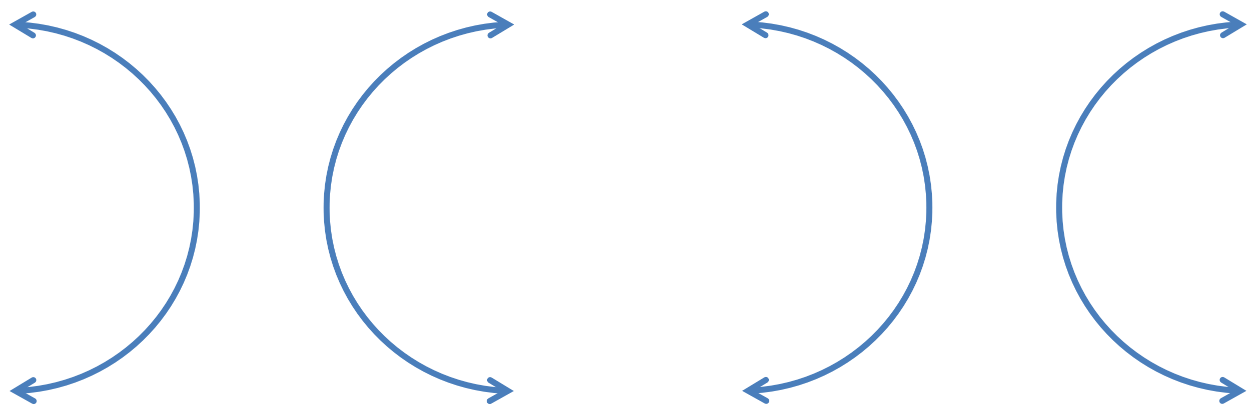
EXAMPLE 2: Consider $16x^2 + 25y^2 - 64x + 150y + 279 = 0$.

Find the center, foci, vertices, and eccentricity of the ellipse and sketch its graph.



EXAMPLE 3: Find an equation of the ellipse with vertices $(0,3)$ and $(8,3)$ and eccentricity $\frac{3}{4}$.

A **hyperbola** is the set of all _________________ _________ for which the absolute value of the difference between the distances from two distinct fixed points called ______________ is constant. The line __________________ connecting the vertices is the ________________ ___________, and the _______________ of the transverse axis is the _____________ of the hyperbola.



Theorem: Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center _________ is

Transverse Axis is Horizontal

or

Transverse Axis is Vertical

The vertices are ___ units from the center, and the foci are ____ units from the center with __________________.

Theorem: Asymptotes of a Hyperbola

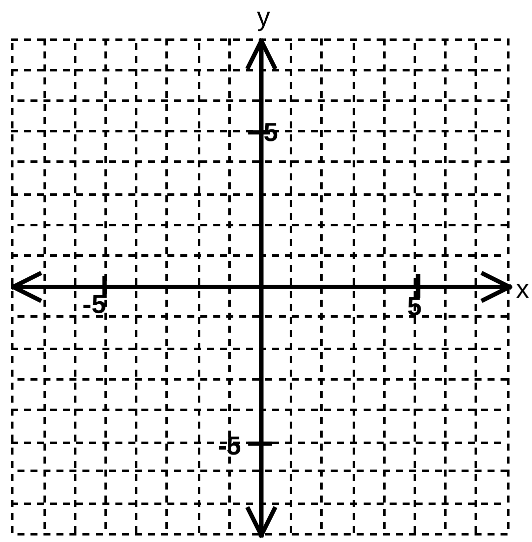
Transverse Axis is Horizontal

or

Transverse Axis is Vertical

EXAMPLE 4: Consider $\dfrac{y^2}{4} - \dfrac{x^2}{2} = 1$.

a. Find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes.



b. Find equations for the tangent lines to the hyperbola at $x = 4$.

c. Find equations for the normal lines to the hyperbola at $x = 4$.

EXAMPLE 4: A cable of a suspension bridge is suspended in the shape of a parabola between two towers that are 120 meters apart and 20 meters above the roadway. The cable touches the roadway midway between the two towers.

a. Find an equation for the parabolic shape of the cable.

b. Find the length of the cable.

10.2: Plane Curves and Parametric Equations

When you finish your homework you should be able to…

- π Sketch the graph of a curve given by a set of parametric equations.
- π Eliminate the parameter in a set of parametric equations.
- π Find a set of parametric equations to represent a curve.

We currently use a ______________ equation involving _______ variables to represent a _____________. This tells us _____________ an object has been but it doesn't tell us ___________ the object was at a given __________ _______. To determine this _________, we introduce a third variable, ___, called a _________________. Using two equations to represent each ___ and ___ as functions of ___ gives us __________________ ________________.

Definition of a Plane Curve

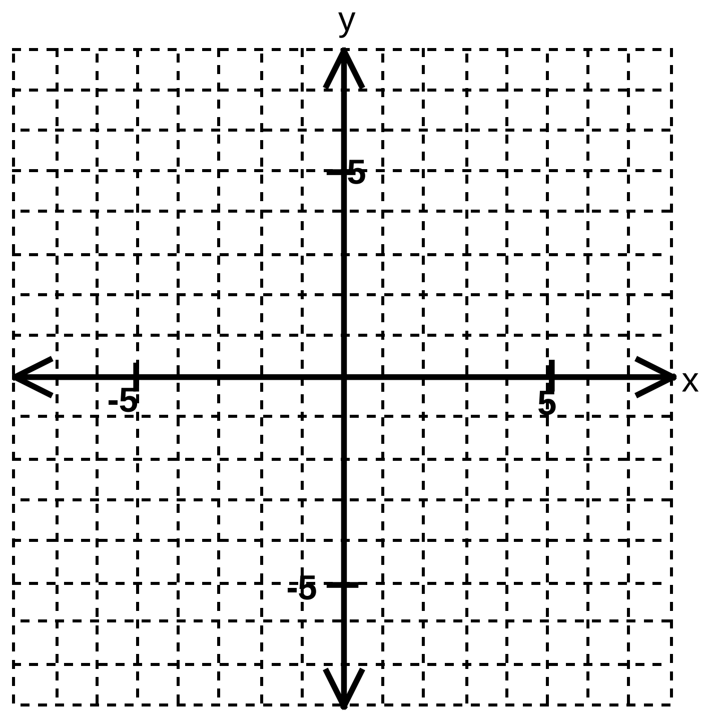
If ___ and ___ are continuous functions of ___ on an interval ___, then the equations

are ________________ equations and ____is the ______________. The set of points _________ obtained as ____ varies over the interval ___ is the ____________ of the parametric equations. Taken together, the _____________ equations and the _________ are a _________ ___________.

EXAMPLE 1: Consider $x = 2t^2$, $y = t^4 + 1$.

a. Sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

t	$x = 2t^2$	$y = t^4 + 1$



b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 2: Consider $x = \cos\theta$, $y = 2\sin 2\theta$.

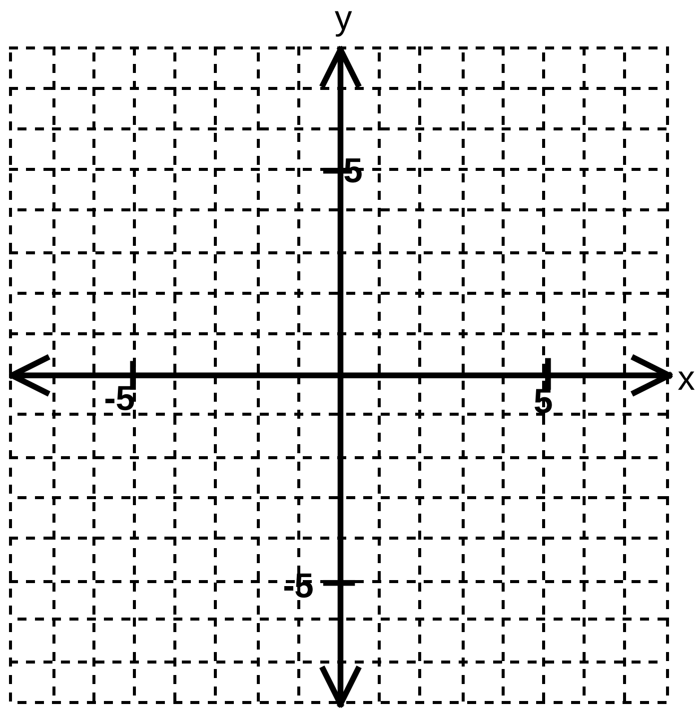
a. Use your graphing calculator to sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 3: Consider $x=-2+3\cos\theta$, $y=-5+3\sin\theta$.

a. Sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

θ	$x=-2+3\cos\theta$	$y=-5+3\sin\theta$



b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 4: Consider $x = e^{2t}$, $y = e^{t}$.

a. Use your graphing calculator to sketch the curve represented by the parametric equations. Be sure to indicate the orientation.

b. Write the corresponding rectangular equation by eliminating the parameter.

EXAMPLE 5: Find a set of parametric equations for the line or conic.

a. Circle: Center $(-6, 2)$, radius 4

b. Ellipse: Vertices $(4, 7)$, $(4, -3)$, Foci: $(4, 5)$, $(4, -1)$.

10.3: Plane Curves and Parametric Equations

When you finish your homework you should be able to…

- π Find the slope of a line tangent to a plane curve.
- π Find the arc length of a plane curve.
- π Find the area of a surface of revolution given in parametric form.

Theorem: Parametric Form of the Derivative

If a smooth curve *C* is given by the equations

then the slope of *C* at _____ is

EXAMPLE 1: Consider $x = 4\cos t$, $y = 2\sin t$, $0 < t < 2\pi$.

a. Find $\frac{dy}{dx}$.

b. Find $\frac{d^2y}{dx^2}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Determine the open t-intervals on which the curve is concave downward or concave upward.

Theorem: Arc Length in Parametric Form

If a smooth curve C is given by the equations ___________ and ___________such that C does not intersect itself on the interval ______________, except possibly at the endpoints, then the arc length of C over the interval is given by

NOTE: Make sure that the arc length is _____________ only once on the interval!!!

EXAMPLE 2: Find the arc length of the curve given by the equations $x = \arcsin t$ and $y = \ln\sqrt{1-t^2}$ on the interval $0 \le t \le \frac{1}{2}$.

Theorem: Area of a Surface of Revolution

If a smooth curve *C* is given by the equations ___________ and ___________such that *C* does not intersect itself on the interval ______________, then the area *S* of the surface of revolution formed by revolving *C* about the coordinate axes is given by

Revolution about the *x*-axis; ____________

Revolution about the *y*-axis; ____________

EXAMPLE 3: Find the area of the surface generated by revolving the curve given by the equations $x = 5\cos\theta$ and $y = 5\sin\theta$ on the interval $0 \le \theta \le \pi$ about the *y*-axis.

EXAMPLE 4: A portion of a sphere of radius *r* is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the cone.

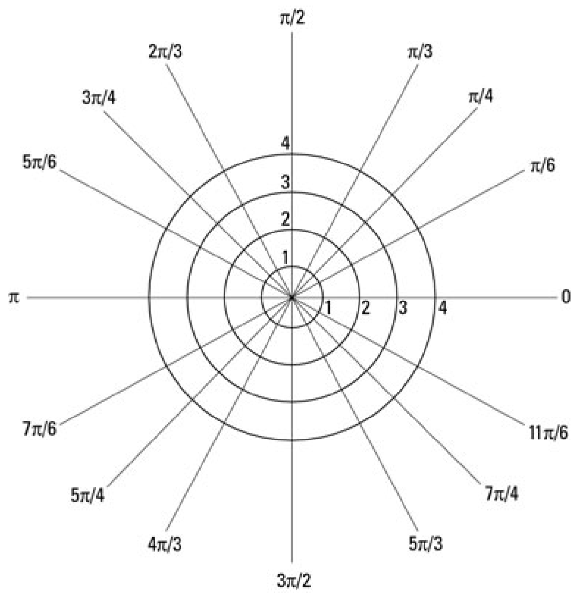
10.4: Polar Coordinates and Graphs

When you finish your homework you should be able to…

- π Convert between rectangular and polar coordinates.
- π Sketch the graph of an equation in polar form.
- π Find the slope of a line tangent to the pole.
- π Identify special polar graphs.

Up to this point, we've been using the _________________ coordinate system to sketch graphs. Now we will be using the ____________ coordinate system to sketch graphs given in _____________ form. This form is very useful in the third semester calculus course as it makes many definite _________________ easier to evaluate after switching from rectangular to polar coordinates. The polar coordinate system has a fixed point *O*, called the _________ or ____________. From the pole, an initial _________ is constructe. This is called the __________ axis. Each point *P* in the plane is assigned ___________ coordinates in the form __________. ___ represents the _____________ distance from ___ to ___ and ____ is the _____________ angle which is _________________________________ from the polar axis to the segment_____. Unlike rectangular coordinates, each point in polar coordinates does NOT have a ____________ representation. Can you figure out another point in polar coordinates which would be equivalent to

(r,θ)? ________________ How about $(-r,\theta+\pi)$? __________________



In general, the point (r,θ) can be written as

where ___ is any integer. The pole is represented by ______, where ___ is any angle.

Theorem: Coordinate Conversion

The polar coordinates (r,θ) of a point are related to the rectangular coordinates ___________ of the point as follows:

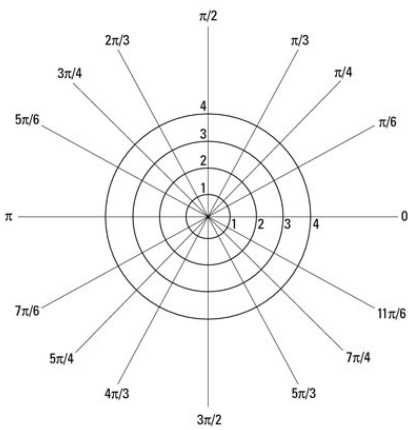
Polar-to-Rectangular

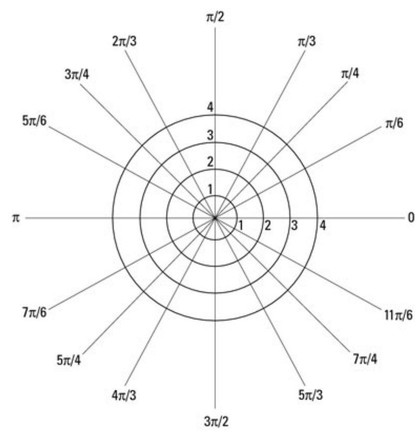
Rectangular-to-Polar

EXAMPLE 1: Plot the point in polar coordinates and find the corresponding rectangular coordinates for the point.

a. $\left(3, \frac{\pi}{4}\right)$

b. $\left(-2, \frac{5\pi}{3}\right)$





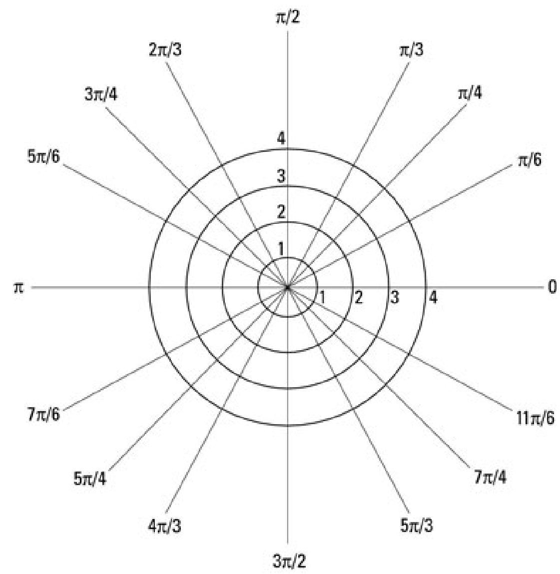
EXAMPLE 2: Find two corresponding polar coordinates for the point given in rectangular coordinates.

a. $\left(3, \frac{\pi}{4}\right)$

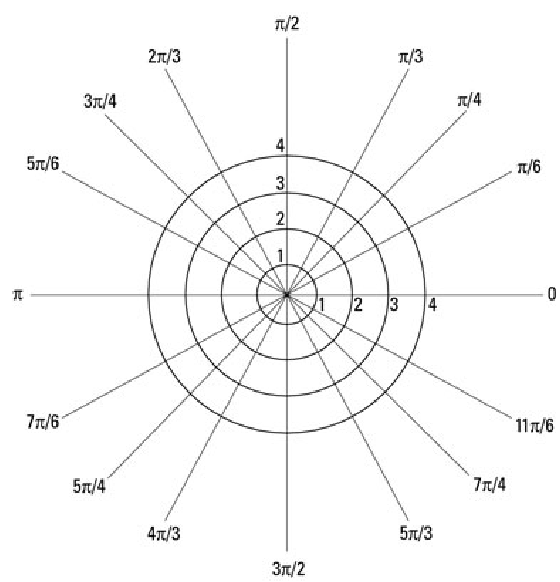
b. $\left(-6, \frac{\pi}{2}\right)$

EXAMPLE 3: Sketch the graph of the polar equation, and convert to rectangular form.

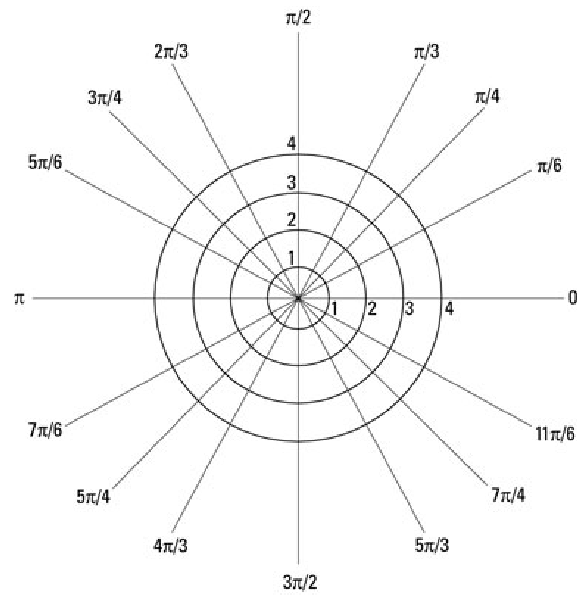
a. $r = -4$



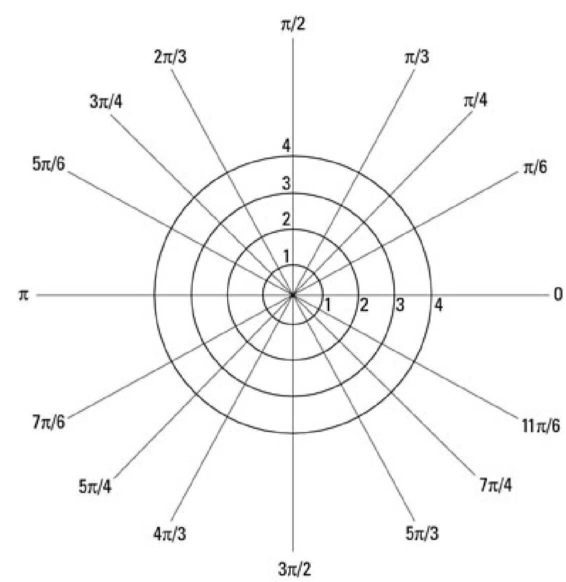
b. $\theta = \dfrac{5\pi}{6}$



c. $r = 3\sin\theta$



d. $r = \cot\theta\csc\theta$



EXAMPLE 4: Convert the rectangular equation to polar form.

a. $x^2 - y^2 = 9$

b. $xy = 4$

Consider $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$.

Theorem: Slope in Polar Form

If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r,θ) is

provided that ________________ at ___________.

HMMMMM…I guess that means…

Solutions of $\frac{dy}{d\theta} = 0$ **yield** ___________________ **tangents, provided** $\frac{dx}{d\theta} \neq 0$.

Solutions of $\frac{dx}{d\theta} = 0$ **yield** ___________________ **tangents, provided** $\frac{dy}{d\theta} \neq 0$.

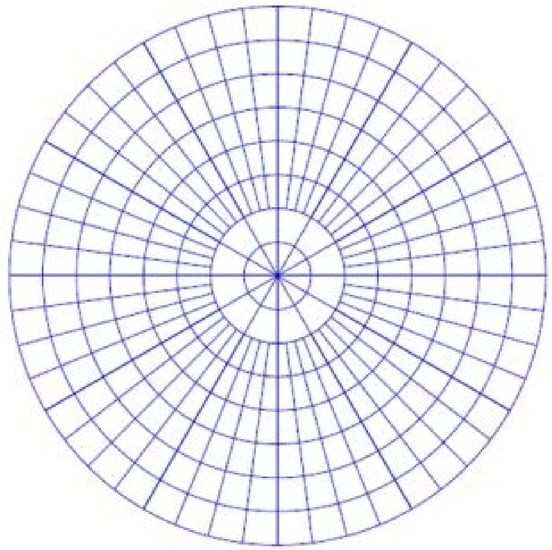
If $\frac{dy}{d\theta} = 0$ **and** $\frac{dx}{d\theta} = 0$ **simultaneously, then no conclusion can be drawn about** ____________ **lines.**

Theorem: Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $f(\alpha) = 0$ is _______________ at the _________ to the graph of ____________.

EXAMPLE 5: Consider $r = 2(1 - \sin\theta)$. Hint: use $\frac{\pi}{24}$ for the increment between the values of θ.

a. Sketch the graph of the equation.



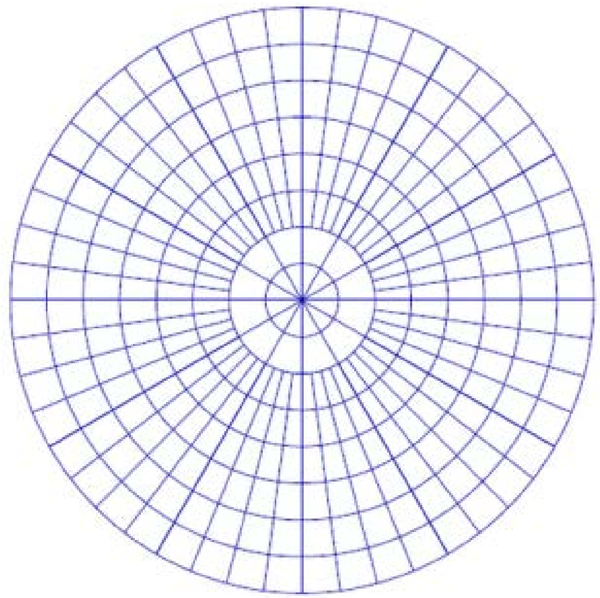
b. Find $\frac{dy}{dx}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Find the tangents at the pole.

EXAMPLE 6: Consider $f(\theta) = 8\cos 3\theta$.

a. Graph the equation by hand.



b. Find $\dfrac{dy}{dx}$.

c. Find all points (if any) of horizontal and vertical tangency to the curve.

d. Find the tangents at the pole.

10.5: *Area and Arc Length in Polar Coordinates*

When you finish your homework you should be able to…

- π Find the points of intersection between polar graphs.
- π Find the area of a region bounded by a polar graph.
- π Find the arc length of a polar graph.
- π Find the area of a surface of revolution (polar form)

To find the points of ____________________ of polar graphs, you merely

__________ the ____________ of ________________ equations.

EXAMPLE 1: Find the points of intersection of the graphs of the equations $r = 3(1+\sin\theta)$ and $r = 3(1-\sin\theta)$.

The formula for the ___________ of a ___________ region is developed by

_______________ infinitely many __________ of __________. Recall that the

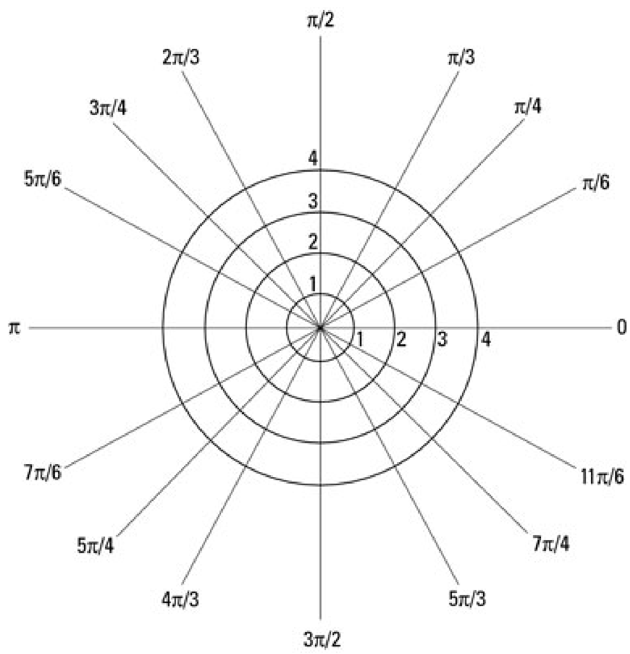
area of a sector is ________________.

Theorem: Area in Polar Coordinates

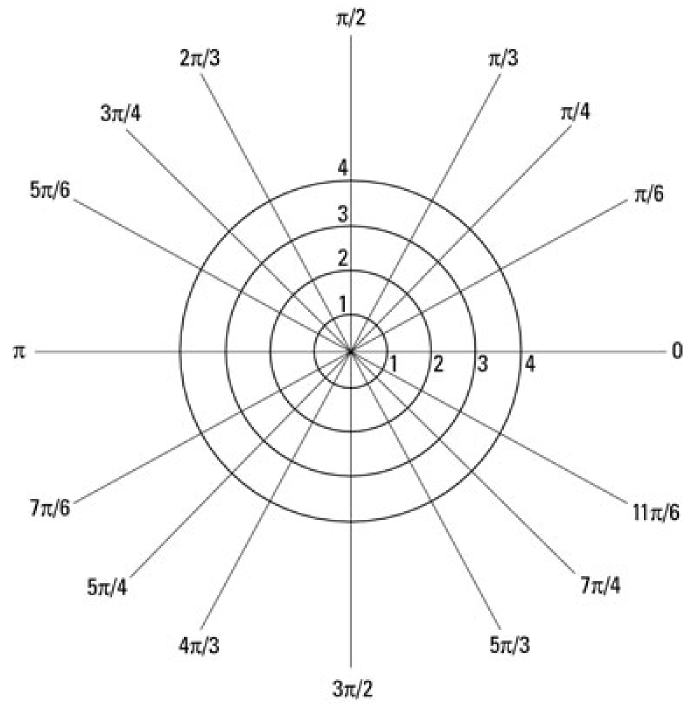
If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$0 < \beta - \alpha \leq 2\pi$

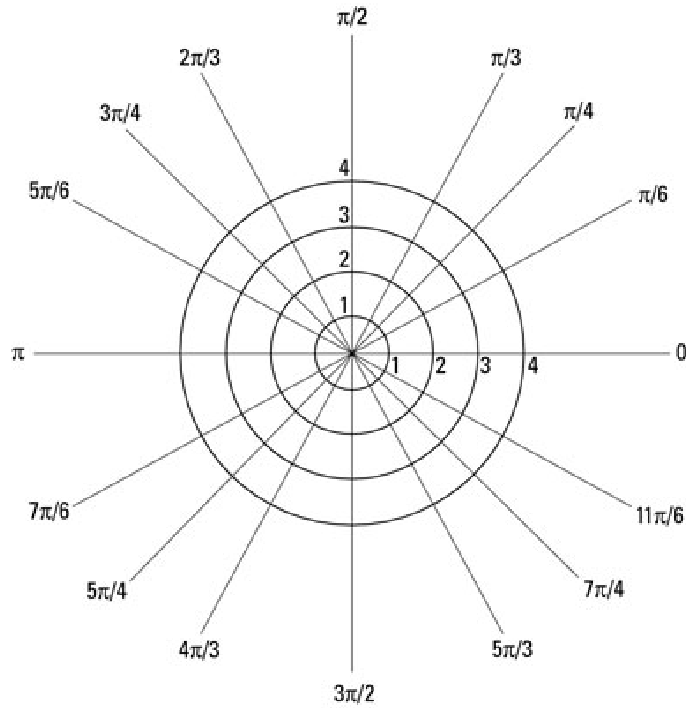
EXAMPLE 2: Find the area of the region of one petal of $r = 4\sin 3\theta$.



EXAMPLE 3: Find the area of the region of the interior of $r = 4 - 4\cos\theta$.



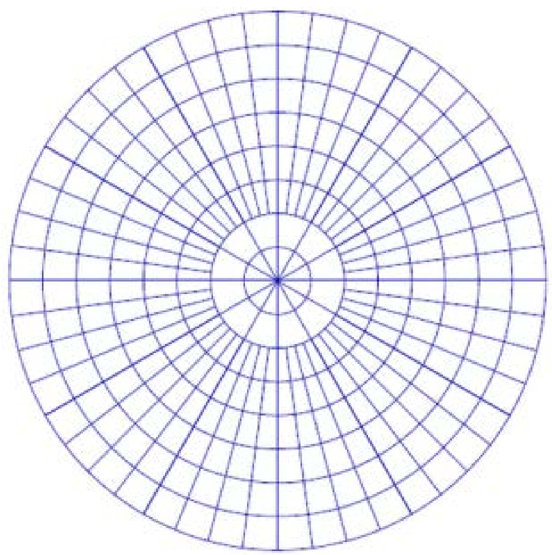
EXAMPLE 4: Find the area of the common interior of $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$.



Theorem: Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

EXAMPLE 5: Find the arc length of the curve $r = 8(1+\cos\theta)$ over the interval $0 \le \theta \le 2\pi$.



Theorem: Area of a Surface of Revolution

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about

the polar axis is:

the line $\theta = \frac{\pi}{2}$ is:

EXAMPLE 6: Find the area of the surface formed by revolving the curve $r = 6\cos\theta$ about the polar axis over the interval $0 \leq \theta \leq \frac{\pi}{2}$.